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Resonances for Hamiltonians with a delta perturbation in one dimension

Claudio Fernández[1], **Guillermo Palma**[2] **and Humberto Prado**[3]

[1] Facultad de Matemática, Pontificia Universidad Católica de Chile, Código postal: 6904441, Santiago, Chile
[2] Departamento de Física, Universidad de Santiago de Chile, Casilla 347-Correo-2, Santiago, Chile
[3] Departamento de Matemática, Universidad de Santiago de Chile, Casilla 307-Correo-2, Santiago, Chile

E-mail: cfernand@mat.puc.cl, gpalma@usach.cl and hprado@usach.cl

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Abstract

We analyse the existence of almost exponentially decaying states associated with quasi-stationary states of the Hamiltonian $H_\omega = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \omega\delta_a$ defined on $L^2(\mathbb{R}_+)$, where δ_a is the repulsive delta potential. We use Krein's formula to study the time evolution of the system defined by H_ω. In this paper we find that the quasi-stationary states in the infinite limit $\omega \to \infty$, decay almost exponentially; this fact can be explained physically due to the existence of resonances.

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1. Introduction

The concepts of resonance and quasi-stationary states play a fundamental role in quantum mechanics. They allow us to understand decaying processes and the associated concept of mean lifetime (see [2, 13, 16]) which we want to address in this paper by means of a one-dimensional (1D) system. We should remark that the existence of quasi-stationary states with almost exponential decay in quantum-mechanical systems has been illustrated in the literature [8, 17]; however, to our knowledge, a rigorous mathematical proof of the existence of such solutions was still missing. The scope of this paper is to study the almost exponential decay of the unitary group associated with a self-adjoint operator with a *delta perturbation*. To this end, we consider the self-adjoint realization of the operator,

$$H_\omega = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \omega\delta_a, \tag{1.1}$$

on the Hilbert space $L^2[0, \infty)$, with Dirichlet boundary conditions at $x = 0$, i.e., $\psi(0) = 0$; this ensures that the particle does not pass to the region $x < 0$. Hamiltonian (1.1) enjoys two important properties, on one side it has an analytic expression for the resolvent, $(z - H_\omega)^{-1}$, which is given by Krein's formula (see section 2). On the other side, it allows an intuitive physical picture for the quasi-stationary states, the resonances and exponential decay, without the cumbersome analysis of more realistic experimental processes, like the positive muon decay μ^+ described in [14], and the so-called beta decay of unstable nuclei such as uranium-238 and the decay of doubly ionized uranium, among many others (see, e.g., [1], and the references therein). Furthermore, the operator (1.1) represents a physical system consisting of a quantum-mechanical particle moving on the positive half-line under the action of a repulsive point interaction at $x = a$ of strength ω. The interaction $\omega\delta_a$ acts as a thin barrier which cannot trap the particle in the interval $[0, a]$, but which for large values of ω leads to a resonance which appears as a trapped state $\psi \in L^2(0, \infty)$ with a large lifetime. We shall prove the existence of states which decay with an approximately exponential rate and which are initially localized on the fixed interval $[0, a]$. We also provide explicit estimates of this decay rate.

From a semiclassical point of view we can introduce the concepts of mean lifetime τ and quasi-stationary states. We want to use a semiclassical analysis to estimate the mean lifetime associated with a quasi-stationary state represented by a particle with energy $E_n = \hbar^2 k_n/2m$, moving in the region $0 \leqslant x \leqslant a$. Such a state becomes a stationary state in the zero penetrability limit $\omega \to \infty$. Its mean lifetime can be obtained as follows. First we compute the transmission coefficient T_n at $x = a$. A straightforward canonical computation leads to:

$$T_n = \left(1 + \left(\frac{1}{k_n a \epsilon}\right)^2\right)^{-1} \approx (k_n a \epsilon)^2$$

for $0 < \epsilon \ll 1$. Now we recall the probabilistic interpretation of quantum mechanics and imagine instead of one quasi-trapped particle in $[0, a]$, there are N such particles which do not interact with each other. As a consequence of the particles hitting the delta barrier from the left and the fact that the transmission coefficient is different from zero; some of them will leave the well by tunnelling. In order to define the decay probability we also need to estimate the frequency of such hits at $x = a$. This can be obtained by the semi-classical relation $v_n/2a$, where $v_n = \hbar k_n/m$ is the particle velocity and $2a$ is the distance between two such consecutive hits. Then we can use the relation for the decay probability [8], that is, decay probability per second equals frequency of hits times penetrability. Now, if we denote by dN the infinitesimal variation of the number N of quasi-trapped particles, then the decay law is given by d$N = -N$ times decay probability per second times d$t = -N/\tau_n$, where the mean lifetime τ_n becomes

$$\tau_n = \frac{2a}{v_n} T^{-1} \approx 2\frac{m^3 a \omega^2}{\hbar^5 k_n^3}.$$

The semiclassical estimation of τ_n relies on the classical expression for v_n, the frequency of collisions. Nevertheless, in order to compute v_n one should construct wave packets which are localized in space within a region much smaller than the scale a, and thus one would require that the wave packets spread significantly over the lifetime. This of course cannot be fulfilled, in particular, for small values of n (see [13], and the references therein), leading to an overestimation of the mean lifetime, when compared to the more accurate results obtained in theorem 5.8.

There are results on approximately exponential decay behaviour for other models. For instance we recall that in [10–12] it is shown that exponential behaviour exists for the class of states obtained by truncating a resonant solution in a region containing the support of the potential. This problem has also been studied by [9] and [15]. On the other hand, the usual analytic dilation methods, employed for example by [7], also give rates of exponential time decay. However, in the context of this paper, as well as in some of the previous works on the subject, analyticity does not hold (see, e.g., [9, 11, 12]).

We also remark that the question about the self-adjointness of Hamiltonians involving the delta distribution has already been studied by several authors; for example, in [3] it has been shown that the perturbation $\omega\delta_a$ belongs to the Kato class $\mathcal{K}_1$ (see also [4]). Even so, for the sake of completeness in the preliminaries below we give a rigorous definition of the operator H_ω.

Our approach consists in studying the corresponding stationary problem, which is determined by the behaviour of the Hamiltonians resolvent near the real axis. Then in section 2 we use the Krein formula to study this problem, in section 3 we show some applications to the one-point perturbations, and in section 4 we state our main results; the proofs of the theorems of section 4 are given in the appendix.

2. Preliminaries

Let $C_c^1(\mathbb{R}_+)$ be the space of the once differentiable functions with compact support contained in $\mathbb{R}_+ =: [0, \infty)$, and let $\mathcal{H}^1 := \mathcal{H}^1(\mathbb{R}_+)$ be the Sobolev space consisting of all the classes of functions $f \in L^2(\mathbb{R}_+)$ for which there is $g \in L^2(\mathbb{R}_+)$ such that $\int_0^\infty f\varphi' = -\int_0^\infty g\varphi$ for all $\varphi \in C_c^1(\mathbb{R}_+)$. Then, $\mathcal{H}^1$ turns into a Hilbert space when endowed with the inner product

$$\langle f, g\rangle_{\mathcal{H}^1} = \langle f, g\rangle_{L^2} + \langle f', g'\rangle_{L^2}.$$

We let $\mathcal{H}_0^1$ be the closure of $C_c^1(\mathbb{R}_+)$ in $\mathcal{H}^1$. As usual, we denote by $\mathcal{H}^{-1}$ the dual space of $\mathcal{H}_0^1$. Then, we have the continuous inclusions with dense images $\mathcal{H}_0^1 \subset L^2(\mathbb{R}_+) \subset \mathcal{H}^{-1}$. We also recall that for every $f \in \mathcal{H}_0^1$ there is a continuous function φ (in the equivalence class of f) such that $f = \varphi$ (a.e.). Then we define

Definition 2.1. *Given $a \in \mathbb{R}_+$, the functional τ_a is defined as $\tau_a(\varphi) = \varphi(a)$, with domain $D(\tau_a) = \mathcal{H}_0^1$, in $L^2(\mathbb{R})$.*

Proposition 2.2. *The functional τ_a enjoys the following properties:*

(1) τ_a is a bounded functional on $\mathcal{H}_0^1$.
(2) If δ_a denotes the delta distribution of $\mathcal{H}^{-1}$, that is $\delta_a(\varphi) = \varphi(a)$, then the dual functional $\tau_a^* : \mathbb{C} \to \mathcal{H}^{-1}$ is given by $\tau_a^*(z) = z\delta_a$.
(3) The operator $\tau_a^*\tau_a : \mathcal{H}_0^1 \to \mathcal{H}^{-1}$ is given by

$$(\tau_a^*\tau_a)(\varphi) = \varphi(a)\delta_a.$$

Proof. We note that there is a C such that $|\varphi(a)| \leqslant \|\varphi\|_\infty \leqslant C\|\varphi\|_{\mathcal{H}_0^1}$ and hence the proof of the theorem follows as a direct consequence of the definition. □

Remark 2.3.

(i) We introduce the *delta operator* from $L^2(\mathbb{R}_+)$ to $\mathcal{H}^{-1}$, to be

$$(\tau_a^*\tau_a)(\varphi) = \varphi(a)\delta_a \qquad \left(\varphi \in \mathcal{H}_0^1\right),$$

and when no confusion arises, we shall denote the operator $\tau_a^*\tau_a$ by δ_a.

(ii) Moreover, under the above considerations we also introduce the Hamiltonian H_ω with a delta operator perturbation on $L^2(\mathbb{R}_+)$ by

$$H_\omega = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \omega\delta_a, \qquad (\omega \in \mathbb{R})$$

with the domain $D(H_\omega) = \{\varphi \in \mathcal{H}_0^1 : \varphi(0) = 0\}$ and taking values in $\mathcal{H}^{-1}$.

(iii) We note that the free Hamiltonian $H_0 =: -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$, with the domain $\mathcal{H}_0^1$, acts on the same spaces as the delta operator.

(iv) Since $\tau_a^*(z) = z\delta_a$ for all $z \in \mathbb{C}$, henceforth we shall identify τ_a^* with the delta distribution $\delta_a \in \mathcal{H}^{-1}$.

3. Krein's formula

In an abstract setting we let $\mathcal{H}$ be a Hilbert space, and suppose that H_0 is a closed operator acting on $\mathcal{H}$ and let $H_\omega = H_0 + \omega A^*B$ where $A, B \in \mathcal{H}^*$ and $\omega \in \mathbb{C}$. If $R(z, H_\omega) := (z - H_\omega)^{-1}$ denotes the resolvent operator of H, then we obtain.

Theorem 3.4. *Let H_0 be a closed operator on a Hilbert space $\mathcal{H}$ and let $A, B \in \mathcal{H}^*$, not necessarily bounded and such that $D(H_0) \subseteq D(B)$. Then the following assertions are verified.*

(i) $BR(z, H_0)A^* : \mathbb{C} \to \mathbb{C}$ whenever $z \in \rho(H_0)$.

(ii) If $H_\omega := H_0 + \omega A^*B$, for some $\omega \in \mathbb{C}$, is such that $\rho(H_\omega) = \rho(H_0)$, then

$$R(z, H_\omega) = R(z, H_0) + \omega k(z)R(z, H_0)A^*BR(z, H_0)$$

for all $z \in \rho(H_0)$, where $k(z) = (1 - \omega BR(z, H_0)A^*)^{-1}$.

Proof. Since (i) is clear, then we proceed to show (ii). First we note that

$$R(z, H_\omega) - R(z, H_0) = \omega R(z, H_\omega)A^*BR(z, H_0). \tag{3.1}$$

Now by (i) we have

$$\omega BR(z, H_0)A^* : \mathbb{C} \to \mathbb{C}.$$

Then we define $k(z) := (1 - \omega BR(z, H_0)A^*)^{-1}$ for z not a singularity. On the other hand we have that $R(z, H_\omega)A^*[1 - \omega BR(z, H_0)A^*] = R(z, H_0)A^*$ and hence

$$R(z, H_\omega)A^* = k(z)R(z, H_0)A^*. \tag{3.2}$$

Now inserting (3.2) into (3.1) it follows that

$$R(z, H_\omega) = R(z, H_0) + \omega k(z)R(z, H_0)A^*BR(z, H_0)$$

and the proof is now finished. □

Corollary 3.5 Perturbation by a rank-1 projection. *Let H_0 be a self-adjoint operator on a Hilbert space $\mathcal{H}$, and let $\varphi_0 \in \mathcal{H}$ such that $\|\varphi_0\| = 1$. If $H_w = H_0 + w(\langle|\varphi_0\rangle\langle\varphi_0|\rangle)$ for some $w \in \mathbb{C}$, then*

$$R(z, H_w)\psi = R(z, H_0)\psi + \left(\frac{w\langle\varphi_0, R(z, H_0)\psi\rangle}{1 - w\langle\varphi_0, R(z, H_0)\varphi_0\rangle}\right) R(z, H_0)\varphi_0 \tag{3.3}$$

for all $\psi \in \mathcal{H}$.

Proof. If we define $A\psi = \langle\varphi_0, \psi\rangle$ then $A^*z = z\varphi_0$ $(z \in \mathbb{C})$. Thus $A^*A\psi = \langle\varphi_0, \psi\rangle\varphi_0$ and hence we write

$$H_w = H_0 + wA^*A.$$

Then the k function corresponding to the Krein formula in this case is given by

$$k(z) = (1 - w\langle\varphi_0, R(z, H_0)\varphi_0\rangle)^{-1}.$$

Moreover, for $\psi \in \mathcal{H}$,

$$R(z, H_0)A^*AR(z, H_0)\psi = \langle\varphi_0, R(z, H_0)\psi\rangle R(z, H_0)\varphi_0$$

and Krein's formula (3.3) follows for a rank-1 perturbation. □

Next we show how to apply Krein's theorem to self-adjoint operators H_0 with a delta perturbation on $L^2([0, \infty))$.

Corollary 3.6 Perturbation by a delta operator**.** *Let H_0 be a closed operator on $L^2(\mathbb{R})$ with $D(H_0) = \mathcal{H}_0^1$ and let $H = H_0 + \omega\delta_a$ for $\omega \in \mathbb{R}$ and $a \in \mathbb{R}$. Then H has domain equal to $\mathcal{H}_0^1$ and takes values in $\mathcal{H}^{-1}$. Moreover, if $\rho(H_0) = \rho(H)$, then $k(z) = (1 - \omega\tau_a R(z, H_0)\tau_a^*)^{-1}$ and*

$$R(H, z) = R(H_0, z) + \omega k(z)R(H_0, z)\tau_a\tau_a^* R(H_0, z). \tag{3.4}$$

4. An application

We now consider the delta perturbation of $H_0 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ acting on $L^2(\mathbb{R}_+)$, that is for $w \in \mathbb{C}, a \in \mathbb{R}_+$

$$H_w = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + w\tau_a^*\tau_a \qquad \text{with} \quad D(H_w) = \left\{\varphi \in \mathcal{H}_0^1 : \varphi(0) = 0\right\}.$$

To compute $R(z, H_\omega)$ first we note that $\tau_a^*\tau$ is defined in terms of δ_a by remark 2.3. Thus by corollary 3.5 we obtain that

$$k(z) = (1 - \omega\tau_a R(z, H_0)\tau_a^*)^{-1}. \tag{4.1}$$

Now for $x, y > 0$ and $\mathrm{Im}\sqrt{z} > 0$ we let

$$K(x, y; z) := \frac{-1}{2\mathrm{i}\sqrt{z}}(\mathrm{e}^{\mathrm{i}\sqrt{z}|x+y|} - \mathrm{e}^{\mathrm{i}\sqrt{z}|x-y|})$$

be the Green function associated with the solution of the Schrödinger equation

$$\begin{cases} -u''(x) + \omega u(x) = f(x) & (x \geqslant 0) \\ u(x) \sim 0, & \text{as } x \to +\infty. \end{cases} \tag{4.2}$$

We recall that

$$R(z, H_0)\psi(x) = \int_0^\infty K(x, y; z)\psi(y)\,\mathrm{d}y \qquad \psi \in L^2(\mathbb{R}_+).$$

Since $R(z, H_0) : \mathcal{H}^{-1} \to \mathcal{H}_0^1$ then $(R(z, H_0)\delta_a)(x) = K(x, a; z)$. Thus

$$\tau_a R(z, H_0)\tau_a^* = K(a, a; z) = \frac{-1}{2\mathrm{i}\sqrt{z}}(\mathrm{e}^{\mathrm{i}2a\sqrt{z}} - 1)$$

and hence by (4.1) we obtain that

$$k(z) = [1 - \omega K(a, a; z]^{-1}.$$

Then $R(z, H_\omega)$ is obtained by applying (3.4). Since

$$(R(z, H_0)\tau_a\tau_a^* R(H_0, z))\psi(x) = \left[\int_0^\infty K(a, y; z)\psi(y)\,\mathrm{d}y\right] K(x, a; z),$$

Krein's identity for H_w yields that

$$R(z, H_w)\psi(x) = \int_0^\infty \left[\frac{K(x, y; z)}{K(x, a; z)} + \frac{\omega K(a, y; z)}{1 - \omega K(a, a; z)}\right]\psi(y)\,\mathrm{d}y K(x, a; z).$$

We summarize these observations in the next theorem.

Theorem 4.7. *Let $H_\omega = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \omega\delta_a$, acting on $\mathcal{H}^\infty$. Then for all $z \in \rho(H_\omega)$ and $\psi \in L^2(\mathbb{R}_+)$*

$$R(z, H_w)\psi(x) = \int_0^\infty \left[\frac{K(x, y; z)}{K(x, a; z)} + \frac{\omega K(a, y; z)}{1 - \omega K(a, a; z)}\right]\psi(y)\,\mathrm{d}y K(x, a; z)$$

where

$$K(x, y; z) := \frac{-1}{2\mathrm{i}\sqrt{z}}(\mathrm{e}^{\mathrm{i}\sqrt{z}|x+y|} - \mathrm{e}^{\mathrm{i}\sqrt{z}|x-y|})$$

for $x, y > 0$ and $\operatorname{Im}\sqrt{z} > 0$ is the Green function of equation (4.2).

5. Existence of almost exponentially decaying states

In this section we show the existence of states which decay with an approximately exponential rate and which are initially localized on the fixed interval $[0, a]$. To this end, we compute $\langle\psi, \mathrm{e}^{\mathrm{i}tH_\omega}\psi\rangle$, for ψ which vanishes outside $[0, a]$.

First, we introduce some preliminary notation. Let $\psi \in L^2(\mathbb{R}_+)$ be such that $\operatorname{supp}(\psi) \subseteq [0, a]$ and define

$$r_\psi(\lambda) := 2\sqrt{\lambda}\left[\int_0^a \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}}\psi(x)\,\mathrm{d}x\right]^2.$$

Henceforth we denote r_ψ simply by r and its Fourier transform by $\widehat{r}(t)$.

We are now ready to state the main result for this section.

Theorem 5.8. *Let ψ be a real-valued function with support contained in $[0, a]$, and $\psi \in L^2(\mathbb{R}_+)$. For each integer n let $\lambda_n = \left(\frac{n\pi}{a}\right)^2$, $\Gamma_{\epsilon,n}^2 = \epsilon\frac{n^3\pi^3}{a^4}$ where $\epsilon = \frac{1}{a\omega}$. Let $z_n = \lambda_n - \Gamma_{\varepsilon,n} - \mathrm{i}\Gamma_{\epsilon,n}$. Then for each $0 < \epsilon < 1$*

$$\langle\psi, \mathrm{e}^{\mathrm{i}tH_\omega}\psi\rangle = C(\Gamma_{\epsilon,n}, \psi)\,\mathrm{e}^{-\mathrm{i}(\lambda_n - \Gamma_{\epsilon,n})t - \Gamma_{\epsilon,n}t} + (\sqrt{\pi/2})\widehat{r}(t),$$

where

$$C(\Gamma_{\epsilon,n}, \psi) = \frac{2r(z_n)}{a^2(1+\mathrm{i})}\left[\frac{\cosh(2a\sqrt{z_n})}{\mathrm{i}\Gamma_{\epsilon,n}} - 2\pi\Gamma_{\epsilon,n}\,\mathrm{e}^{2\mathrm{i}a\sqrt{z_n}}\right].$$

Previous to the next corollary we should remark that if $\psi_{2n}(x) = \sin\left(\frac{2n\pi}{a}x\right)$ for $0 \leqslant x \leqslant a$, and $\psi_{2n}(x) = 0$ elsewhere, then

$$r_{2n}(\lambda) = \frac{8n^2\pi^2}{a_2}\frac{1}{\sqrt{\lambda}}\frac{\sin^2(a\sqrt{\lambda})}{(\lambda - 4n^2\pi^2/a^2)^2}. \tag{5.1}$$

Now under the notation of the above theorem we have the following estimate for $\widehat{r}(t)$.

Corollary 5.9. *Let* $\psi_{2n}(x) = \sin\left(\frac{2n\pi}{a}x\right)$ *for* $0 \leqslant x \leqslant a$*, and* $\lambda_{2n} = \left(\frac{2n\pi}{a}\right)^2$*. Then*

(i)

$$\lim_{\lambda \to \lambda_{2n}} r(\lambda) = 2n\pi.$$

(ii) For $0 < \epsilon \ll 1$

$$|C(\Gamma_{\epsilon,n}, \psi_{2n})| \approx \frac{2}{\epsilon}\left(\frac{\cosh(4n\pi)}{\sqrt{2n\pi}}\right).$$

(iii) For $0 < \epsilon \ll 1$ *there is a* $T_\epsilon > 0$ *such that*

$$|\widehat{r}(t)| \leqslant |C(\Gamma_{\epsilon,n}, \psi_{2n})| \qquad (0 \leqslant t \leqslant T_\epsilon).$$

6. Concluding remarks

(i) We have shown the existence of almost exponentially decaying states in the context of 1D quantum mechanics. In particular we have considered a particle of unitary mass moving in the positive half-axis in the presence of a delta potential at $x = a$, with penetrability ω. Using the exact expression due to Krein for the resolvent operator for this problem, which is also known from the path integral formalism [5, 6], we demonstrate that for initially trapped states, described by a function with support in $[0, a]$, whose wave numbers κ_n are close enough to the resonance value $k_n = n\pi/a$, the probability of remaining trapped at time t decays exponentially in the zero penetrability limit when $\omega \to \infty$. The lifetime associated with them is given by $\tau_n = (1/\sqrt{\epsilon})\sqrt{ak_n}$, with $\Gamma_{\epsilon,n} = \sqrt{\epsilon}/\sqrt{ak_n}$.

(ii) In the case that the initial state is of finite support but is not almost stationary we conjecture that under these conditions there is a polynomial type of decay instead of almost exponential.

(iii) In dimension 3 one should expect a similar result for the decay of the quasi-stationary states, which corresponds to wavefunctions with radial dependence proportional to the spherical Bessel functions. This extension can be carried on along the lines outlined by Holstein [8].

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Appendix

We recall that

$$\langle\psi, \mathrm{e}^{\mathrm{i}tH_\omega}\psi\rangle = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}t\lambda}\,\mathrm{d}\langle\psi, E_\lambda\psi\rangle$$

where E_λ is the spectral measure for H_ω. Since H_ω has an absolutely continuous spectrum in $(0, \infty)$, we have, for any interval $I \subseteq (0, \infty)$,

$$\int_I \mathrm{e}^{\mathrm{i}t\lambda}\,\mathrm{d}\langle\psi, E_\lambda\psi\rangle = \frac{1}{\pi}\lim_{\delta\to 0^+}\int_I \mathrm{e}^{-\mathrm{i}\lambda t}\,\mathrm{Im}\langle\psi, R(\lambda + \mathrm{i}\delta, H_\omega)\psi\rangle\,\mathrm{d}\lambda.$$

In what follows we assume that $\psi \in L^2(\mathbb{R}_+)$ is such that $\mathrm{supp}(\psi) \subseteq [0, a]$. Then we define

$$G_1(\psi)(\lambda) := \int_0^a \psi(x) \int_0^a K(x, y; \lambda)\psi(y)\, \mathrm{d}y\, \mathrm{d}x$$

and

$$G_2(\psi)(\lambda) := \int_0^a \psi(x) \int_0^a \left(\frac{K(a, y; z)K(x, a; \lambda)}{\frac{1}{w} - K(a, a; \lambda)} \right) \psi(y)\, \mathrm{d}y\, \mathrm{d}x.$$

Then by the Krein formula, theorem 4.7, it follows that

$$\langle \psi, R(\lambda, H_\omega)\psi \rangle = G_1(\psi)(\lambda) + G_2(\psi)(\lambda). \tag{A.1}$$

Proof of theorem 5.8. We first compute $\langle \psi, R(\lambda, H_w)\psi \rangle$. By the properties of $K(x, y; z)$ when $\mathrm{Im}\,\sqrt{z} > 0$ it follows that

$$\frac{1}{\pi} \lim_{\delta \to 0^+} \int_0^\infty \mathrm{e}^{-\mathrm{i}\lambda t}\, \mathrm{Im}\,\langle \psi, R(\lambda + \mathrm{i}\delta, H_\omega)\psi \rangle\, \mathrm{d}\lambda = \frac{1}{\pi} \int_0^\infty \mathrm{e}^{-\mathrm{i}\lambda t}\, \mathrm{Im}\langle \psi, R(\lambda, H_\omega)\psi \rangle\, \mathrm{d}\lambda.$$

Next we recall that

$$r(\lambda) := 2\sqrt{\lambda} \left[\int_0^a \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}} \psi(x)\, \mathrm{d}x \right]^2.$$

Then it easily follows that $\mathrm{Im}\, G_1(\psi)(\lambda) = \frac{r(\lambda)}{2}$. Thus

$$\int_0^\infty \mathrm{e}^{-\mathrm{i}t\lambda}\, \mathrm{Im}\, G_1(\psi)(\lambda)\, \mathrm{d}\lambda = \frac{1}{\sqrt{2}} \widehat{r}(t), \tag{A.2}$$

where $\widehat{r}$ stands for the Fourier transform of r. Hence it remains to find $\int_0^\infty \mathrm{e}^{-\mathrm{i}t\lambda}\, \mathrm{Im} G_2(\psi)(\lambda)\, \mathrm{d}\lambda$. We first recall that $K(a, a; \lambda) = \frac{-1}{2\mathrm{i}\sqrt{\lambda}}(\mathrm{e}^{\mathrm{i}2\sqrt{\lambda}a} - 1)$, and

$$G_2(\psi)(\lambda) = \left(\frac{2\mathrm{i}\sqrt{\lambda}\, \mathrm{e}^{\mathrm{i}2a\sqrt{\lambda}}}{-1 + 2\mathrm{i}\varepsilon a\sqrt{\lambda} + \mathrm{e}^{\mathrm{i}2a\sqrt{\lambda}}} \right) \left[\int_0^a \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}} \psi(x)\, \mathrm{d}x \right]^2. \tag{A.3}$$

Next we denote

$$f_\epsilon(\lambda) := -1 + 2\mathrm{i}\varepsilon a\sqrt{\lambda} + \mathrm{e}^{\mathrm{i}2a\sqrt{\lambda}} \qquad \text{for} \quad \varepsilon := \frac{1}{aw}.$$

Thus by (A.3) we have

$$G_2(\psi)(\lambda) = \mathrm{i}\mathrm{e}^{\mathrm{i}2a\sqrt{\lambda}} \frac{r(\lambda)}{f_\epsilon(\lambda)}$$

where

$$f_\epsilon(\lambda) = (-1 + \cos(2a\sqrt{\lambda}) + \mathrm{i}(\varepsilon a\sqrt{\lambda}) + \sin(2a\sqrt{\lambda})).$$

Now if we let $u = \sqrt{\lambda}$, and $u_n = \sqrt{\lambda_n} = \frac{n\pi}{a}$, then $\cos(a(u - u_n)) = \cos(au)$ and $\sin(a(u - u_n)) = \sin(au)$. Thus we obtain that

$$f_\epsilon(u^2) = (1 + \cos(2a(u - u_n)) + \mathrm{i}(\varepsilon au + \sin(2a(u - u_n))). \tag{A.4}$$

Now by taking the Taylor expansion of $\cos(u)$ and $\sin(u)$ respectively by (A.4) we arrive at

$$f_\epsilon(u^2) = -\frac{(2a(u - u_n))^2}{2} + 0((u - u_n)^2) + \mathrm{i}(\epsilon au + 2a(u - u_n) + 0(u - u_n)). \tag{A.5}$$

Since $u = \sqrt{\lambda} = \sqrt{\lambda_n}\sqrt{1 + \frac{\Delta\lambda}{\lambda_n}} + \mathrm{o}(\lambda)$, where $\Delta\lambda = \lambda - \lambda_n$ and $\lambda > 0$, it follows that

$$\sqrt{\lambda} = \sqrt{\lambda_n}\left(1 + \frac{\Delta\lambda}{2\lambda_n}\right) + \mathrm{o}(\lambda), \qquad (\lambda > 0).$$

Since $u - u_n = \sqrt{\lambda} - \sqrt{\lambda_n}$, then $u - u_n = (\lambda - \lambda_n)\frac{1}{2\sqrt{\lambda_n}} + \mathrm{o}(\lambda)$, and by (A.5) we arrive at

$$f_\epsilon(\lambda) = \frac{-a^2}{2}\left[(\lambda - \lambda_n)^2 - \mathrm{i}2\Gamma_{\varepsilon,n}^2\right] + D(\lambda) \tag{A.6}$$

where $D(\lambda)$ is an analytic function such that when $z_n = \lambda_n - \Gamma_{\epsilon,n} - \mathrm{i}\Gamma_{\epsilon,n}$ then $\lim_{z\to z_n} D(z) = f(z_n)$. On the other hand we have the decomposition

$$\begin{aligned} G_2(\psi)(\lambda) &= \frac{2\mathrm{i}\,\mathrm{e}^{2\mathrm{i}\sqrt{\lambda}a} r(\lambda)}{-a^2\big((\lambda - \lambda_n)^2 - 2\mathrm{i}\Gamma_{\epsilon,n}^2\big)} + \frac{2\mathrm{i}\,\mathrm{e}^{2\mathrm{i}\sqrt{\lambda}a} D(\lambda) r(\lambda)}{a^2\big((\lambda - \lambda_n)^2 - 2\mathrm{i}\Gamma_{\epsilon,n}^2\big) f_\varepsilon(\lambda)} \\ &= a(\lambda) + b(\lambda), \end{aligned}$$

where

$$a(\lambda) := \frac{2\mathrm{i}\,\mathrm{e}^{2\mathrm{i}a\sqrt{\lambda}} r(\lambda)}{-a^2\big((\lambda - \lambda_n)^2 - 2\mathrm{i}\Gamma_{\epsilon,n}\big)} \qquad \text{and} \qquad b(\lambda) := \frac{2\mathrm{i}\,\mathrm{e}^{2\mathrm{i}\sqrt{\lambda}a} D(\lambda) r(\lambda)}{a^2\big((\lambda - \lambda_n)^2 - 2\mathrm{i}\Gamma_{\epsilon,n}^2\big) f_\varepsilon(\lambda)}.$$

Now

$$\operatorname{Im} a(\lambda) = \frac{2\cos(2a\sqrt{\lambda}) r(\lambda)(\lambda - \lambda_n)^2 - 4\Gamma_\varepsilon^2 \sin(2a\sqrt{\lambda}) r(\lambda)}{-a^2\big((\lambda - \lambda_n)^4 + \Gamma_{\epsilon,n}^4\big)}.$$

Moreover, we have that

$$g(z) = \frac{\left[2\cos(2a\sqrt{\lambda}) r(z)(z - \lambda_n)^2 - 4\Gamma_\varepsilon^2 \sin(2a\sqrt{z}) R(z)\right]\mathrm{e}^{-\mathrm{i}zt}}{-a^2(z - (\lambda_n - \Gamma_{\epsilon,n} - \mathrm{i}\Gamma_{\epsilon,n}))(z - (\lambda_n + \Gamma_{\epsilon,n} + \mathrm{i}\Gamma_{\epsilon,n}))}$$

has a simple pole at $z_n = \lambda_n - \Gamma_{\epsilon,n} - \mathrm{i}\Gamma_{\epsilon,n}$. Hence

$$\int_0^\infty \mathrm{e}^{-\mathrm{i}\lambda t} \operatorname{Im}(a(\lambda))\,\mathrm{d}\lambda = -\frac{4\pi\Gamma_{\epsilon,n}}{a^2(1+\mathrm{i})}\,\mathrm{e}^{2\mathrm{i}a\sqrt{z_n}} r(z_n)\,\mathrm{e}^{-\mathrm{i}z_n t}. \tag{A.7}$$

Next we compute $\int_0^\infty \mathrm{e}^{-\mathrm{i}\lambda t} \operatorname{Im} b(\lambda)\,\mathrm{d}\lambda$. First we note that for small values of ϵ, $f_\varepsilon(z) = -1 + \mathrm{i}\varepsilon a\sqrt{z} + \mathrm{e}^{2\mathrm{i}a\sqrt{z}}$ does not vanish for z close enough to $z_n = \lambda_n - \Gamma_\varepsilon - \mathrm{i}\Gamma_{\epsilon,n}$. Thus $\frac{1}{f(z)}$ has no singularities in a neighbourhood of z_n. Then by applying the same reasoning as in the computation of integral (A.7) we obtain

$$\int_0^\infty \mathrm{e}^{-\mathrm{i}\lambda t} b(\lambda)\,\mathrm{d}\lambda = \frac{2\pi}{a^2}\,\frac{\mathrm{e}^{2\mathrm{i}a\sqrt{z_n}} r(z_n)\,\mathrm{e}^{-\mathrm{i}z_n t}}{(1+\mathrm{i})\Gamma_{\epsilon,n}}. \tag{A.8}$$

Now $\lim_{z\to z_n} D(z) = f(z_n)$ by (A.6). Next we recall that $\operatorname{Im} b(\lambda) = \frac{1}{2\mathrm{i}}(b(\lambda) - \overline{b}(\lambda))$. Moreover,

$$\overline{b}(\lambda)) = \frac{-2\mathrm{i}\,\mathrm{e}^{-2\mathrm{i}a\sqrt{\lambda}}\overline{D}(\lambda) r(\lambda)}{a^2\left[(\lambda - \lambda_n)^2 + 2\mathrm{i}\Gamma_\varepsilon^2\right]\overline{f}_\varepsilon(\lambda)}$$

Hence

$$\int_0^\infty \mathrm{e}^{-\mathrm{i}\lambda t}\overline{b}(\lambda)\,\mathrm{d}\lambda = \left(\frac{-2\pi\,\mathrm{e}^{-2\mathrm{i}a\sqrt{z_n}} r(z_n)}{a^2(1+\mathrm{i})\Gamma_\varepsilon}\right)\mathrm{e}^{-\mathrm{i}z_n t}. \tag{A.9}$$

Thus by (A.8) and (A.9) we arrive at

$$\int_0^\infty \mathrm{e}^{-\mathrm{i}\lambda t} \operatorname{Im} b(\lambda)\,\mathrm{d}\lambda = \frac{2\pi r(z_n)}{a^2\mathrm{i}(1+\mathrm{i})\Gamma_{\epsilon,n}} \cosh(2a\sqrt{z_n})\,\mathrm{e}^{-\mathrm{i}z_n t}. \tag{A.10}$$

Thus from (A.7) and (A.10) we obtain

$$\frac{1}{\pi}\int_0^\infty \mathrm{e}^{-\mathrm{i}\lambda t}\,\mathrm{Im}\,G_2(\psi)(\lambda)\,\mathrm{d}\lambda = C(\Gamma_{\epsilon,n},\psi)\,\mathrm{e}^{-\mathrm{i}(\lambda_n-\Gamma_{\epsilon,n})t-\Gamma_{\epsilon,n}t} \tag{A.11}$$

where

$$C(\Gamma_{\epsilon,n},\psi) = \frac{2r(z_n)}{a^2(1+\mathrm{i})}\left[\frac{\cosh(2\sqrt{z_n}a)}{\mathrm{i}\Gamma_{\epsilon,n}} - 2\pi\,\Gamma_{\epsilon,n}\,\mathrm{e}^{2\mathrm{i}a\sqrt{z_n}}\right].$$

Hence by (A.2) and (A.11) we arrive at

$$\frac{1}{\pi}\int_0^\infty \mathrm{e}^{-\mathrm{i}\lambda t}\,\mathrm{Im}\langle\psi, R(\lambda, H_\omega)\psi\rangle\,\mathrm{d}\lambda = C(\Gamma_{\epsilon,n},\psi)\,\mathrm{e}^{-\mathrm{i}(\lambda_n-\Gamma_{\epsilon,n})t-\Gamma_{\epsilon,n}t} + \frac{1}{\pi\sqrt{2}}\widehat{r}(t),$$

and the proof is finished. □

Proof of corollary 4.9. The proofs (i) and (ii) are clear. Now statement (iii) follows from the fact that

$$\widehat{r}(0) = \frac{16n^2\pi^2}{a^2\sqrt{2\pi}}\int_0^\infty \frac{\sin^2(au)}{(u^2-4n^2\pi^2/a^2)^2}\,\mathrm{d}u.$$

Thus given $0 < \epsilon \ll 1$, there is an interval $[0, T_\epsilon]$ such that (ii) together with the continuity of $\widehat{r}(t)$ implies the truth of assertion (iii). □

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